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FRACTIONAL HERMITE-HADAMARD TYPE INEQUALITIES FOR n-TIMES r-CONVEX FUNCTIONS

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ABSTRACT. In this paper, we establish some Hermite-Hadamard type inequalities for function whose n^{th} derivatives are r-convex via Riemann-Liouville integral operators.

1. INTRODUCTION

One of the most well-known inequalities in mathematics for convex functions is the so called Hermite-Hadamard integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \qquad (1.1)$$

where f is a real convex function on the finite interval $[a,b]$. If the function f is concave, then (1.1) holds in the reverse direction (see [4]).

The Hermite-Hadamard inequality play an important role in nonlinear analysis, and optimization. The above

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double inequality has attracted many researchers, various generalizations, refinements, extensions and variants of (1.1) have appeared in the literature, via classical integration and fractional calculus.

In [1] Dragomir et al. gave the following results

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{8}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right)^{\frac{p-1}{p}}.$$

In [6] Pearce et al. established the following result

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

In [8] Sarikaya et al. investigate the following fractional analogue of the result given in [1]

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}(J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a))\right|$$
$$\leq \frac{(2^{\alpha}-1)(b-a)}{2^{\alpha+1}(\alpha+1)}\left(\left|f'(a)\right|+\left|f'(b)\right|\right).$$

The main purpose of this paper is to establish a new Hermite-Hadamard type inequalities for functions whose n^{th} derivatives are r-convex via Riemann-Liouville fractional integral operators.

2. Preliminaries

In this sections we recall some definitions and lemmas, we assume that I is an interval of $\mathbb{R}$

Definition 1. [7] *A function $f : I \to \mathbb{R}$ is said to be convex, if*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 2. [7] *A positive function $f : I \to \mathbb{R}$ is said to be logarithmically convex, if*

$$f(tx + (1-t)y) \le [f(x)]^t [f(y)]^{1-t}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 3. [5] *A positive function $f : I \subseteq [0,\infty) \to \mathbb{R}$ is said to be r-convex on I, if*

$$f(tx+(1-t)y) \le \begin{cases} [tf^r(x) + (1-t)f^r(y)]^{\frac{1}{r}}, & r \neq 0 \\ [f(x)]^t [f(y)]^{1-t}, & r = 0 \end{cases}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 4. [2, 3] *Let $f \in L[a,b]$. The Riemann-Liouville fractional integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \ge 0$ are defined by*

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \qquad x > a$$

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \qquad b > x$$

respectively. Where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$, is the Gamma function and $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.

Lemma 1. [4] *For $a \geq 0$ and $b \geq 0$, the following algebraic inequalities are true*

$$(a+b)^{\lambda} \leq 2^{\lambda-1}\left(a^{\lambda}+b^{\lambda}\right), \qquad \text{for } \lambda \geq 1$$

and

$$(a+b)^{\lambda} \leq a^{\lambda}+b^{\lambda}, \qquad \text{for } 0 \leq \lambda \leq 1.$$

Lemma 2. [9] *For $\alpha > 0$ and $k > 0$, $z > 0$,*

$$\begin{aligned} J(\alpha,k) &= \int_0^1 (1-t)^{\alpha-1} k^t dt = \sum_{i=1}^{\infty} \frac{(\ln k)^{i-1}}{(\alpha)_i} < \infty \\ H(\alpha,k,z) &= \int_0^z t^{\alpha-1} k^t dt = z^{\alpha} k^z \sum_{i=1}^{\infty} \frac{(-z\ln k)^{i-1}}{(\alpha)_i} < \infty, \end{aligned}$$

where $(\alpha)_i = \prod_{j=0}^{i-1} (\alpha+j)$.

Lemma 3. [10] *Let $n \in \mathbb{N}$ and $\alpha > 0$, and let $f : [a,b] \subset \mathbb{R}_0 \to \mathbb{R}$ be an n-times differentiable function on (a,b). If $f^{(n)} \in L([a,b])$, then we have*

$$\begin{aligned} &\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}} (J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)) \\ &= \sum_{k=0}^{n-1} \tfrac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)} \left(f^{(k)}(a) + (-1)^k f^{(k)}(b)\right) \\ &- \frac{(b-a)^n}{2} \int_0^1 \left((-1)^{n-1}(1-t)^{\alpha+n-1} - t^{\alpha+n-1}\right) \\ &\times f^{(n)}(at+(1-t)b)\,dt. \end{aligned}$$

3. Main results

Theorem 1. *Let $f:[a,b]\subset\mathbb{R}_0\to\mathbb{R}$ be an n-times differentiable function on (a,b) such that $f^{(n)}\in L\left([a,b]\right)$ with $f^{(n)}(a)\neq 0$ and $f^{(n)}(b)\neq 0$. If $\left|f^{(n)}\right|$ is r-convex with $r\geq 0$, then the following fractional inequalities holds*

if n is odd and $r>0$

$$\begin{aligned}&\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right)\right.\\&\left.-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|\\&\leq\tfrac{(b-a)^n c_r}{2}\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right)\left(2B_{\frac{1}{2}}\left(\alpha+n,\frac{1}{r}+1\right)\right.\\&\left.+\ \tfrac{2^{\alpha+n+\frac{1}{r}-1}-1}{\left(\alpha+n+\frac{1}{r}\right)2^{\alpha+n+\frac{1}{r}-1}}-B\left(\frac{1}{r}+1,\alpha+n\right)\right),\end{aligned}$$

if n is even and $r>0$

$$\begin{aligned}&\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right)\right.\\&\left.-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|\\&\leq\tfrac{(b-a)^n c_r}{2}\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right)\\&\times\left(\tfrac{1}{\alpha+n+\frac{1}{r}}+B\left(\tfrac{1}{r}+1,\alpha+n\right)\right),\end{aligned}$$

if $r=0$ and $\left|f^{(n)}(a)\right|=\left|f^{(n)}(b)\right|$

$$\begin{aligned}&\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right)\right.\\&\left.-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|\end{aligned}$$

$$\leq \begin{cases} \frac{2^{\alpha+n-1}-1}{(\alpha+n)2^{\alpha+n-1}} (b-a)^n \left|f^{(n)}(b)\right| & \text{if } n \text{ is odd} \\ \frac{(b-a)^n}{2(\alpha+n)} \left(\left|f^{(n)}(b)\right| + \left|f^{(n)}(b)\right|\right) & \text{if } n \text{ is even,} \end{cases}$$

if $r=0$ *and* $\left|f^{(n)}(a)\right| \neq \left|f^{(n)}(b)\right|$ *and* n *is odd*

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}(J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)) - \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)} \left(f^{(k)}(a) + (-1)^k f^{(k)}(b)\right)\right|$$

$$\leq \frac{(b-a)^n \sqrt{\left|f^{(n)}(a)\right|\left|f^{(n)}(b)\right|}}{2} \times \left(\left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{\frac{1}{2}} \sum_{i=1}^{\infty} \frac{\left(\ln \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{i-1}}{(\alpha+n)_i} - \frac{1}{2^{\alpha+n-1}} \sum_{i=1}^{\infty} \frac{\left(\frac{1}{2}\ln \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{i-1}}{(\alpha+n)_i} - \frac{1}{2^{\alpha+n-1}} \sum_{i=1}^{\infty} \frac{\left(\frac{1}{2}\ln \frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{i-1}}{(\alpha+n)_i} + \left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{\frac{1}{2}} \sum_{i=1}^{\infty} \frac{\left(\ln \frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{i-1}}{(\alpha+n)_i} \right),$$

if $r=0$ *and* $\left|f^{(n)}(a)\right| \neq \left|f^{(n)}(b)\right|$ *and* n *is even*

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}(J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)) - \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)} \left(f^{(k)}(a) + (-1)^k f^{(k)}(b)\right)\right|$$

$$\leq \frac{(b-a)^n}{2}\left(\left|f^{(n)}(b)\right|\sum_{i=1}^{\infty}\frac{\left(\ln\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{i-1}}{(\alpha+n)_i}\right.$$

$$\left.+\left|f^{(n)}(a)\right|\sum_{i=1}^{\infty}\frac{\left(-\ln\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{i-1}}{(\alpha+n)_i}\right),$$

where

$$c_r=\begin{cases}2^{\frac{1}{r}-1} & \text{if } 0<r<1\\ 1 & \text{if } r\geq 1,\end{cases} \tag{3.1}$$

$B(.,.)$ and $B_x(.,.)$ are beta and incomplete beta functions respectively.

Proof. From Lemma 3, and properties of modulus, we have

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right)\right.$$
$$\left.-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|$$
$$=\frac{(b-a)^n}{2}\left|\int_0^1\left((-1)^{n-1}(1-t)^{\alpha+n-1}-t^{\alpha+n-1}\right)\right.$$
$$\left.\times f^{(n)}(at+(1-t)b)\,dt\right|$$
$$\leq\frac{(b-a)^n}{2}\int_0^1\left|(-1)^{n-1}(1-t)^{\alpha+n-1}-t^{\alpha+n-1}\right|$$
$$\times\left|f^{(n)}(at+(1-t)b)\right|dt. \tag{3.2}$$

Clearly, if n is an even number

$$\left|(-1)^{n-1}(1-t)^{\alpha+n-1}-t^{\alpha+n-1}\right|=(1-t)^{\alpha+n-1}+t^{\alpha+n-1}, \tag{3.3}$$

and if n is an odd number, we have

$$\begin{aligned}&\left|(-1)^{n-1}(1-t)^{\alpha+n-1}-t^{\alpha+n-1}\right|\\ =&\left\{\begin{array}{ll}(1-t)^{\alpha+n-1}-t^{\alpha+n-1} & \text{if } 0\leq t<\frac{1}{2}\\ t^{\alpha+n-1}-(1-t)^{\alpha+n-1} & \text{if } \frac{1}{2}\leq t\leq 1.\end{array}\right.\end{aligned} \tag{3.4}$$

Assume that $r>0$, and n is an even number, using (3.2), (3.3), and r-convexity of $\left|f^{(n)}\right|$, we get

$$\begin{aligned}&\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right)\right.\\ &\left.-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right)\right|\\ \leq\;&\tfrac{(b-a)^{n}}{2}\left(\int_{0}^{1}(1-t)^{\alpha+n-1}\right.\\ &\times\left[t\left|f^{(n)}(a)\right|^{r}+(1-t)\left|f^{(n)}(b)\right|^{r}\right]^{\frac{1}{r}}dt\\ &\left.+\int_{0}^{1}t^{\alpha+n-1}\left[t\left|f^{(n)}(a)\right|^{r}+(1-t)\left|f^{(n)}(b)\right|^{r}\right]^{\frac{1}{r}}dt\right).\end{aligned} \tag{3.5}$$

From Lemma 1, we have

$$\begin{aligned}&\left[t\left|f^{(n)}(a)\right|^{r}+(1-t)\left|f^{(n)}(b)\right|^{r}\right]^{\frac{1}{r}}\\ &\leq c_{r}\left[t^{\frac{1}{r}}\left|f^{(n)}(a)\right|+(1-t)^{\frac{1}{r}}\left|f^{(n)}(b)\right|\right],\end{aligned} \tag{3.6}$$

where c_r is defined as in (3.1).

Substituting (3.6) in (3.5), we get

$$\begin{aligned}
&\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}(J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a))\right. \\
&\left.-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right)\right| \\
\leq\ & \tfrac{(b-a)^{n}c_{r}}{2} \\
&\times\left(\int_{0}^{1}(1-t)^{\alpha+n-1}\left[t^{\frac{1}{r}}\left|f^{(n)}(a)\right|+(1-t)^{\frac{1}{r}}\left|f^{(n)}(b)\right|\right]dt\right. \\
&\left.+\int_{0}^{1}t^{\alpha+n-1}\left[t^{\frac{1}{r}}\left|f^{(n)}(a)\right|+(1-t)^{\frac{1}{r}}\left|f^{(n)}(b)\right|\right]dt\right) \\
=\ & \tfrac{(b-a)^{n}c_{r}}{2}\left(\left|f^{(n)}(a)\right|\left(\int_{0}^{1}t^{\frac{1}{r}}(1-t)^{\alpha+n-1}dt+\int_{0}^{1}t^{\alpha+n-1+\frac{1}{r}}dt\right)\right. \\
&\left.+\left|f^{(n)}(b)\right|\left(\int_{0}^{1}(1-t)^{\alpha+n-1+\frac{1}{r}}dt+\int_{0}^{1}t^{\alpha+n-1}(1-t)^{\frac{1}{r}}dt\right)\right) \\
=\ & \tfrac{(b-a)^{n}c_{r}}{2} \\
&\times\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right)\left(\tfrac{1}{\alpha+n+\frac{1}{r}}+B\left(\tfrac{1}{r}+1,\alpha+n\right)\right).
\end{aligned}\tag{3.7}$$

In the case where $r>0$ and n is an odd number, using (3.2), (3.4), and r-convexity of $\left|f^{(n)}\right|$ we get

$$\begin{aligned}
&\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}(J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a))\right. \\
&\left.-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(b-a)^n c_r}{2}\left(\int_0^{\frac{1}{2}}\left((1-t)^{\alpha+n-1}-t^{\alpha+n-1}\right)\right.\\
&\quad\times\left[t^{\frac{1}{r}}\left|f^{(n)}(a)\right|+(1-t)^{\frac{1}{r}}\left|f^{(n)}(b)\right|\right]dt\\
&\quad+\int_{\frac{1}{2}}^{1}\left(t^{\alpha+n-1}-(1-t)^{\alpha+n-1}\right)\\
&\quad\left.\times\left[t^{\frac{1}{r}}\left|f^{(n)}(a)\right|+(1-t)^{\frac{1}{r}}\left|f^{(n)}(b)\right|\right]dt\right)\\
&= \frac{(b-a)^n c_r}{2}\left(\left(\int_0^{\frac{1}{2}}t^{\frac{1}{r}}(1-t)^{\alpha+n-1}dt-\int_0^{\frac{1}{2}}t^{\alpha+n-1+\frac{1}{r}}dt\right.\right.\\
&\quad\left.+\int_{\frac{1}{2}}^{1}t^{\alpha+n-1+\frac{1}{r}}dt-\int_{\frac{1}{2}}^{1}t^{\frac{1}{r}}(1-t)^{\alpha+n-1}dt\right)\left|f^{(n)}(a)\right|\\
&\quad+\left|f^{(n)}(b)\right|\\
&\quad\times\left(\int_0^{\frac{1}{2}}(1-t)^{\alpha+n-1+\frac{1}{r}}dt-\int_0^{\frac{1}{2}}t^{\alpha+n-1}(1-t)^{\frac{1}{r}}dt\right.\\
&\quad\left.\left.+\int_{\frac{1}{2}}^{1}t^{\alpha+n-1}(1-t)^{\frac{1}{r}}dt-\int_{\frac{1}{2}}^{1}(1-t)^{\alpha+n-1+\frac{1}{r}}dt\right)\right)\\
&= \frac{(b-a)^n c_r}{2}\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right)\\
&\quad\times\left(2B_{\frac{1}{2}}\left(\tfrac{1}{r}+1,\alpha+n\right)-B\left(\tfrac{1}{r}+1,\alpha+n\right)\right.\\
&\quad\left.+\frac{2^{\alpha+n+\frac{1}{r}}-2}{\left(\alpha+n+\frac{1}{r}\right)2^{\alpha+n+\frac{1}{r}}}\right). \qquad (3.8)
\end{aligned}
$$

Now, suppose that $r=0$, then $\left|f^{(n)}\right|$ is log-convex, we have

$$\left|f^{(n)}\left(at+(1-t)\,b\right)\right| \leq \left|f^{(n)}\left(a\right)\right|^{t}\left|f^{(n)}\left(b\right)\right|^{1-t} = \left|f^{(n)}\left(b\right)\right|\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{t}. \tag{3.9}$$

If n is an even number, from (3.2), (3.3), (3.9) and Lemma 2, we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right) - \sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}\left(a\right)+(-1)^{k}f^{(k)}\left(b\right)\right)\right| \\
&\leq \frac{(b-a)^{n}}{2}\left(\left|f^{(n)}\left(b\right)\right|\int_{0}^{1}(1-t)^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{t}dt + \left|f^{(n)}\left(b\right)\right|\int_{0}^{1}t^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{t}dt\right). \\
&= \begin{cases}
\frac{(b-a)^{n}}{2(\alpha+n)}\left(\left|f^{(n)}\left(b\right)\right|+\left|f^{(n)}\left(b\right)\right|\right) \\
\text{if } \left|f^{(n)}\left(a\right)\right|=\left|f^{(n)}\left(b\right)\right| \\
\frac{(b-a)^{n}}{2}\left(\left|f^{(n)}\left(b\right)\right|\sum_{i=1}^{\infty}\frac{\left(\ln\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{i-1}}{(\alpha+n)_{i}} + \left|f^{(n)}\left(a\right)\right|\sum_{i=1}^{\infty}\frac{\left(-\ln\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{i-1}}{(\alpha+n)_{i}}\right) \\
\text{if } \left|f^{(n)}\left(a\right)\right|\neq\left|f^{(n)}\left(b\right)\right|.
\end{cases}
\end{aligned} \tag{3.10}$$

where we have used

$$\int_0^1 (1-t)^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^t dt=\sum_{i=1}^{\infty}\frac{\left(\ln\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{i-1}}{(\alpha+n)_i}$$

and

$$\int_0^1 t^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^t dt=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\sum_{i=1}^{\infty}\frac{\left(-\ln\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{i-1}}{(\alpha+n)_i}.$$

In the case where n is an odd number, using (3.2), (3.4) and (3.9), we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right)\right.\\
&\left.-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|\\
\leq\ &\frac{(b-a)^n\left|f^{(n)}(b)\right|}{2}\left(\int_0^{\frac{1}{2}}\left((1-t)^{\alpha+n-1}-t^{\alpha+n-1}\right)\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^t dt\right.\\
&\left.+\int_{\frac{1}{2}}^1\left(t^{\alpha+n-1}-(1-t)^{\alpha+n-1}\right)\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^t dt\right)\\
=\ &\frac{(b-a)^n\left|f^{(n)}(b)\right|}{2}\left(\int_0^{\frac{1}{2}}(1-t)^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^t dt\right.\\
&-\int_0^{\frac{1}{2}}t^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^t dt+\int_{\frac{1}{2}}^1 t^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^t dt
\end{aligned}$$

$$-\int_{\frac{1}{2}}^{1}(1-t)^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{t}dt\Bigg). \tag{3.11}$$

If $\left|f^{(n)}(a)\right| = \left|f^{(n)}(b)\right|$, (3.11) gives

$$\begin{aligned}&\Bigg|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a+}^{\alpha}f(b)-J_{b-}^{\alpha}f(a)\right)\\&-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right)\Bigg|\\ \leq\ & \frac{2^{\alpha+n-1}-1}{(\alpha+n)\,2^{\alpha+n-1}}(b-a)^{n}\left|f^{(n)}(b)\right|. \end{aligned}\tag{3.12}$$

If $\left|f^{(n)}(a)\right| \neq \left|f^{(n)}(b)\right|$, (3.11) gives

$$\begin{aligned}&\Bigg|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a+}^{\alpha}f(b)-J_{b-}^{\alpha}f(a)\right)\\&-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right)\Bigg|\\ \leq\ & \tfrac{(b-a)^{n}\left|f^{(n)}(b)\right|}{2}\\&\times\left(\int_{0}^{\frac{1}{2}}\left((1-t)^{\alpha+n-1}-t^{\alpha+n-1}\right)\left(\tfrac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{t}dt\right.\\&\left.+\int_{\frac{1}{2}}^{1}\left(t^{\alpha+n-1}-(1-t)^{\alpha+n-1}\right)\left(\tfrac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{t}dt\right)\\ =\ & \tfrac{(b-a)^{n}\left|f^{(n)}(b)\right|}{2}\left(\int_{0}^{\frac{1}{2}}(1-t)^{\alpha+n-1}\left(\tfrac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{t}dt\right.\end{aligned}$$

$$
\begin{aligned}
& -\int_0^{\frac{1}{2}} t^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^t dt \\
& +\int_{\frac{1}{2}}^{1} t^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^t dt \\
& \left.-\int_{\frac{1}{2}}^{1}(1-t)^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^t dt\right) \\
= \quad & \frac{(b-a)^n\sqrt{\left|f^{(n)}(a)\right|\left|f^{(n)}(b)\right|}}{2} \\
& \times\left(\left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{\frac{1}{2}}\sum_{i=1}^{\infty}\frac{\left(\ln\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{i-1}}{(\alpha+n)_i}\right. \\
& -\frac{1}{2^{\alpha+n-1}}\sum_{i=1}^{\infty}\frac{\left(\frac{1}{2}\ln\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{i-1}}{(\alpha+n)_i} \\
& -\frac{1}{2^{\alpha+n-1}}\sum_{i=1}^{\infty}\frac{\left(\frac{1}{2}\ln\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{i-1}}{(\alpha+n)_i} \\
& \left.+\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{\frac{1}{2}}\sum_{i=1}^{\infty}\frac{\left(\ln\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{i-1}}{(\alpha+n)_i}\right), \quad (3.13)
\end{aligned}
$$

where we have used

$$
\int_0^{\frac{1}{2}}(1-t)^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^t dt \qquad (3.14)
$$

$$=\int_0^1 (1-t)^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^t dt$$

$$-\int_{\frac{1}{2}}^1 (1-t)^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^t dt$$

$$=\int_0^1 (1-t)^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^t dt$$

$$-\frac{1}{2^{\alpha+n}}\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\int_0^1 t^{\alpha+n-1}\left(\left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{\frac{1}{2}}\right)^t dt$$

$$=\sum_{i=1}^{\infty}\frac{\left(\ln\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{i-1}}{(\alpha+n)_i}$$

$$-\frac{1}{2^{\alpha+n}}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{\frac{1}{2}}\sum_{i=1}^{\infty}\frac{\left(\frac{1}{2}\ln\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{i-1}}{(\alpha+n)_i},$$

$$\int_0^{\frac{1}{2}} t^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^t dt \tag{3.15}$$

$$=\frac{1}{2^{\alpha+n}}\int_0^1 t^{\alpha+n-1}\left(\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{\frac{1}{2}}\right)^t dt$$

$$=\frac{1}{2^{\alpha+n}}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{\frac{1}{2}}\sum_{i=1}^{\infty}\frac{\left(\frac{1}{2}\ln\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{i-1}}{(\alpha+n)_i},$$

$$\int_{\frac{1}{2}}^1 t^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^t dt \tag{3.16}$$

$$=\int_0^1 t^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^t dt-\int_0^{\frac{1}{2}} t^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^t dt$$

$$=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\sum_{i=1}^{\infty}\frac{\left(\ln\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{i-1}}{(\alpha+n)_i}$$

$$-\frac{1}{2^{\alpha+n}}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{\frac{1}{2}}\sum_{i=1}^{\infty}\frac{\left(\frac{1}{2}\ln\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{i-1}}{(\alpha+n)_i},$$

and

$$\int_{\frac{1}{2}}^1 (1-t)^{\alpha+n-1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^t dt$$

$$=\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)\int_0^{\frac{1}{2}} t^{\alpha+n-1}\left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^t dt \tag{3.17}$$

$$=\frac{1}{2^{\alpha+n}}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{\frac{1}{2}}\sum_{i=1}^{\infty}\frac{\left(\frac{1}{2}\ln\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{i-1}}{(\alpha+n)_i}.$$

The desired result follows from (3.7), (3.8), (3.10), (3.12), and (3.13). □

Remark 1. *In Theorem 1 if we take* $r=n=1$*, we obtain Theorem 3 from* [8]*. Moreover if we put* $\alpha=1$ *we get Theorem 2.2 from* [1]*.*

Corollary 1. *The fractional trapezoid inequality for differentiable* log*-convex functions is given by the following* *inequality*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right)-\frac{f(a)+f(b)}{2}\right|$$
$$\leq \begin{cases} \frac{2^{\alpha}-1}{(\alpha+1)2^{\alpha}}(b-a)\left|f'(b)\right| \quad \text{if } \lambda=1, \\ \frac{(b-a)\sqrt{|f'(a)||f'(b)|}}{2} \\ \times\left(\lambda^{-\frac{1}{2}}\sum_{i=1}^{\infty}\frac{(\ln\lambda)^{i-1}}{(\alpha+1)_i}-\frac{1}{2^{\alpha}}\sum_{i=1}^{\infty}\frac{\left(\frac{1}{2}\ln\lambda\right)^{i-1}}{(\alpha+1)_i}\right. \\ \left.-\frac{1}{2^{\alpha}}\sum_{i=1}^{\infty}\frac{\left(-\frac{1}{2}\ln\lambda\right)^{i-1}}{(\alpha+1)_i}+\lambda^{\frac{1}{2}}\sum_{i=1}^{\infty}\frac{(-\ln\lambda)^{i-1}}{(\alpha+1)_i}\right) \\ \text{if } \lambda\neq 1, \end{cases}$$

where $\lambda=\frac{|f'(a)|}{|f'(b)|}$.

Theorem 2. *Let $f:[a,b]\subset\mathbb{R}_0\to\mathbb{R}$ be an n-times differentiable function on (a,b) such that $f^{(n)}\in L([a,b])$ with $f^{(n)}(a)\neq 0$ and $f^{(n)}(b)\neq 0$. If $\left|f^{(n)}\right|^q$ is r-convex with $r\geq 0$, such that $q>1$ and $\frac{1}{q}+\frac{1}{p}=1$, then the following fractional inequalities holds*

If n is an odd number and $r>0$

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right)\right.$$
$$\left.-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right)\right|$$
$$\leq \frac{(b-a)^{n}}{2^{\alpha+n+\frac{1}{p}}}\left(\left(\frac{2^{p\alpha+pn-p+1}-1}{p\alpha+pn-p+1}\right)^{\frac{1}{p}}-\left(\frac{1}{p\alpha+pn-p+1}\right)^{\frac{1}{p}}\right)\left(\frac{rc_r}{1+r}\right)^{\frac{1}{q}}$$
$$\times\left(\left(\left(\tfrac{1}{2}\right)^{\frac{1}{r}+1}\left|f^{(n)}(a)\right|^{q}+\left(1-\left(\tfrac{1}{2}\right)^{\frac{1}{r}+1}\right)\left|f^{(n)}(b)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(1-\left(\tfrac{1}{2}\right)^{\frac{1}{r}+1}\right)\left|f^{(n)}(a)\right|^{q}+\left(\tfrac{1}{2}\right)^{\frac{1}{r}+1}\left|f^{(n)}(b)\right|^{q}\right)^{\frac{1}{q}}\right),$$

if n is an even number and $r>0$

$$\begin{aligned}&\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}(J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a))\right.\\&\left.-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right)\right|\\\leq\ &\frac{(b-a)^{n}}{(p\alpha+pn-p+1)^{\frac{1}{p}}}\left(\tfrac{rc_{r}}{1+r}\right)^{\frac{1}{q}}\left(\left|f^{(n)}(a)\right|^{q}+\left|f^{(n)}(b)\right|^{q}\right)^{\frac{1}{q}},\end{aligned}$$

if $r=0$ and $\left|f^{(n)}(a)\right|=\left|f^{(n)}(b)\right|$

$$\begin{aligned}&\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}(J_{a^{+}}^{\alpha}f(b)-J_{b^{-}}^{\alpha}f(a))\right.\\&\left.-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right)\right|\\\leq\ &\begin{cases}\frac{(b-a)^{n}}{2^{\alpha+n}}\left(\left(\frac{2^{p\alpha+pn-p+1}-1}{p\alpha+pn-p+1}\right)^{\frac{1}{p}}-\left(\frac{1}{p\alpha+pn-p+1}\right)^{\frac{1}{p}}\right)\left|f^{(n)}(b)\right|\\ \text{if } n \text{ is odd}\\ \frac{(b-a)^{n}}{(p\alpha+pn-p+1)^{\frac{1}{p}}}\left|f^{(n)}(b)\right|\\ \text{if } n \text{ is even,}\end{cases}\end{aligned}$$

if $r=0$ and $\left|f^{(n)}(a)\right|\neq\left|f^{(n)}(b)\right|$ and n is an odd number

$$\begin{aligned}&\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}(J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a))\right.\\&\left.-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right)\right|\end{aligned}$$

$$\leq \frac{(b-a)^n}{2^{\alpha+n+\frac{1}{p}}}\left(\left(\frac{2^{p\alpha+pn-p+1}-1}{p\alpha+pn-p+1}\right)^{\frac{1}{p}}-\left(\frac{1}{p\alpha+pn-p+1}\right)^{\frac{1}{p}}\right)$$
$$\left(\left|f^{(n)}(b)\right|^{\frac{1}{2}}-\left|f^{(n)}(a)\right|^{\frac{1}{2}}\right)$$
$$\times\left(\left(\frac{\left|f^{(n)}(a)\right|^{\frac{q}{2}}-\left|f^{(n)}(b)\right|^{\frac{q}{2}}}{\ln\left|f^{(n)}(a)\right|^{q}-\ln\left|f^{(n)}(b)\right|^{q}}\right)^{\frac{1}{q}}\right),$$

if $r=0$ *and* $\left|f^{(n)}(a)\right|\neq\left|f^{(n)}(b)\right|$ *and* n *is an even number*

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right)\right.$$
$$\left.-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|$$
$$\leq \frac{(b-a)^n}{(p\alpha+pn-p+1)^{\frac{1}{p}}}\left(\frac{\left|f^{(n)}(a)\right|^q-\left|f^{(n)}(b)\right|^q}{\ln\left|f^{(n)}(a)\right|^q-\ln\left|f^{(n)}(b)\right|^q}\right)^{\frac{1}{q}},$$

where c_r *is defined as in* (3.1)*.*

Proof. From Lemma 3, and properties of modulus, we have

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right)\right.$$
$$\left.-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|$$
$$\leq \frac{(b-a)^n}{2}\int_0^1\left|(-1)^{n-1}(1-t)^{\alpha+n-1}-t^{\alpha+n-1}\right|$$
$$\times\left|f^{(n)}(at+(1-t)b)\right|dt$$

$$= \begin{cases} \frac{(b-a)^n}{2}\int\limits_0^1 \left((1-t)^{\alpha+n-1}+t^{\alpha+n-1}\right) \\ \times \left|f^{(n)}\left(at+(1-t)\,b\right)\right| dt \text{ if } n \text{ is even} \\ \frac{(b-a)^n}{2}\left(\int\limits_0^{\frac{1}{2}} \left((1-t)^{\alpha+n-1}-t^{\alpha+n-1}\right)\right. \\ \times \left|f^{(n)}\left(at+(1-t)\,b\right)\right| dt \\ +\int\limits_{\frac{1}{2}}^1 \left(t^{\alpha+n-1}-(1-t)^{\alpha+n-1}\right) \\ \left.\times \left|f^{(n)}\left(at+(1-t)\,b\right)\right| dt\right) \\ \text{if } n \text{ is odd.} \end{cases} \tag{3.18}$$

Assume that $r > 0$ and n is an even number, from (3.18), and Hölder inequality we have

$$\begin{aligned} &\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right)\right. \\ &\left.-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right| \\ \leq\ &\frac{(b-a)^n}{2}\left(\left(\left(\int_0^1 (1-t)^{p\alpha+pn-p}\right)^{\frac{1}{p}}\right.\right. \\ &\times\left(\int_0^1 \left|f^{(n)}\left(at+(1-t)\,b\right)\right|^q dt\right)^{\frac{1}{q}} \\ &+\left(\int_0^1 t^{p\alpha+pn-p}dt\right)^{\frac{1}{p}} \end{aligned}$$

$$\times \left(\int_0^1 \left| f^{(n)}\left(at+(1-t)b\right) \right|^q dt \right)^{\frac{1}{q}} \Bigg)$$

$$= \frac{(b-a)^n}{(p\alpha+pn-p+1)^{\frac{1}{p}}} \left(\int_0^1 \left| f^{(n)}\left(at+(1-t)b\right) \right|^q dt \right)^{\frac{1}{q}}. \tag{3.19}$$

Using r-convexity of $\left|f^{(n)}\right|^q$ and Lemma 1, we get

$$\begin{aligned}
&\Bigg| \tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}} \left(J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right) \\
&-\sum_{k=0}^{n-1} \tfrac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)} \left(f^{(k)}(a) + (-1)^k f^{(k)}(b) \right) \Bigg| \\
\leq\ & \tfrac{(b-a)^n (c_r)^{\frac{1}{q}}}{(p\alpha+pn-p+1)^{\frac{1}{p}}} \\
&\times \left(\left| f^{(n)}(a) \right|^q \int_0^1 t^{\frac{1}{r}} dt + \left| f^{(n)}(b) \right|^q \int_0^1 (1-t)^{\frac{1}{r}} dt \right)^{\frac{1}{q}} \\
=\ & \tfrac{(b-a)^n}{(p\alpha+pn-p+1)^{\frac{1}{p}}} \left(\tfrac{rc_r}{1+r} \right)^{\frac{1}{q}} \left(\left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q \right)^{\frac{1}{q}},
\end{aligned} \tag{3.20}$$

where c_r is defined by (3.1).

In the case where $r>0$ and n is an odd number, from (3.18) and Hölder inequality we deduce

$$\begin{aligned}
&\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right)\right.\\
&\left.-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right)\right|\\
\leq\ & \tfrac{(b-a)^{n}}{2}\left(\left(\left(\left(\int_{0}^{\frac{1}{2}}(1-t)^{p\alpha+pn-p}\right)^{\frac{1}{p}}-\left(\int_{0}^{\frac{1}{2}}t^{p\alpha+pn-p}dt\right)^{\frac{1}{p}}\right)\right.\right.\\
&\times\left(\int_{0}^{\frac{1}{2}}\left|f^{(n)}(at+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\\
&+\left(\left(\int_{\frac{1}{2}}^{1}t^{p\alpha+pn-p}dt\right)^{\frac{1}{p}}-\left(\int_{\frac{1}{2}}^{1}(1-t)^{p\alpha+pn-p}dt\right)^{\frac{1}{p}}\right)\\
&\left.\left(\int_{\frac{1}{2}}^{1}\left|f^{(n)}(at+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\right)\\
=\ & \tfrac{(b-a)^{n}}{2^{\alpha+n+\frac{1}{p}}}\left(\left(\tfrac{2^{p\alpha+pn-p+1}-1}{p\alpha+pn-p+1}\right)^{\frac{1}{p}}-\left(\frac{1}{p\alpha+pn-p+1}\right)^{\frac{1}{p}}\right)\\
&\times\left(\left(\int_{0}^{\frac{1}{2}}\left|f^{(n)}(at+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^{1}\left|f^{(n)}(at+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\right). \qquad (3.21)
\end{aligned}$$

Now, using r-convexity of $\left|f^{(n)}\right|^q$ and Lemma 1, we get

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left(J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right)\right.\\
&\left.-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|\\
\leq\ & \frac{(b-a)^n(c_r)^{\frac{1}{q}}}{2^{\alpha+n+\frac{1}{p}}}\left(\left(\frac{2^{p\alpha+pn-p+1}-1}{p\alpha+pn-p+1}\right)^{\frac{1}{p}}-\left(\frac{1}{p\alpha+pn-p+1}\right)^{\frac{1}{p}}\right)\\
&\times\left(\left(\left|f^{(n)}(a)\right|^q\int_0^{\frac{1}{2}}t^{\frac{1}{r}}dt+\left|f^{(n)}(b)\right|^q\int_0^{\frac{1}{2}}(1-t)^{\frac{1}{r}}dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\left|f^{(n)}(a)\right|^q\int_{\frac{1}{2}}^1 t^{\frac{1}{r}}dt+\left|f^{(n)}(b)\right|^q\int_{\frac{1}{2}}^1(1-t)^{\frac{1}{r}}dt\right)^{\frac{1}{q}}\right)\\
=\ & \frac{(b-a)^n}{2^{\alpha+n+\frac{1}{p}}}\left(\left(\frac{2^{p\alpha+pn-p+1}-1}{p\alpha+pn-p+1}\right)^{\frac{1}{p}}-\left(\frac{1}{p\alpha+pn-p+1}\right)^{\frac{1}{p}}\right)\left(\frac{rc_r}{1+r}\right)^{\frac{1}{q}}\\
&\times\left(\left(\left(\tfrac{1}{2}\right)^{\frac{1}{r}+1}\left|f^{(n)}(a)\right|^q+\left(1-\left(\tfrac{1}{2}\right)^{\frac{1}{r}+1}\right)\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\left(1-\left(\tfrac{1}{2}\right)^{\frac{1}{r}+1}\right)\left|f^{(n)}(a)\right|^q+\left(\tfrac{1}{2}\right)^{\frac{1}{r}+1}\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right),
\end{aligned}\tag{3.22}$$

where c_r is defined by (3.1). Now, we treat the case where $r=0$.

Clearly we have

$$\begin{aligned}&\left|f^{(n)}\left(at+(1-t)\,b\right)\right|^{q}\leq\left(\left|f^{(n)}\left(a\right)\right|^{q}\right)^{t}\left(\left|f^{(n)}\left(b\right)\right|^{q}\right)^{1-t}\\&=\left|f^{(n)}\left(b\right)\right|^{q}\left(\frac{\left|f^{(n)}(a)\right|^{q}}{\left|f^{(n)}(b)\right|^{q}}\right)^{t}.\end{aligned}\tag{3.23}$$

In the case where n is an even number, combining (3.23) and (3.19) we get

$$\begin{aligned}&\Bigg|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}(J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a))\\&\quad-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}\left(a\right)+(-1)^{k}f^{(k)}\left(b\right)\right)\Bigg|\\&\leq\frac{(b-a)^{n}}{(p\alpha+pn-p+1)^{\frac{1}{p}}}\left(\left|f^{(n)}\left(b\right)\right|^{q}\int_{0}^{1}\left(\frac{\left|f^{(n)}(a)\right|^{q}}{\left|f^{(n)}(b)\right|^{q}}\right)^{t}dt\right)^{\frac{1}{q}}\\&=\begin{cases}\frac{(b-a)^{n}}{(p\alpha+pn-p+1)^{\frac{1}{p}}}\left|f^{(n)}\left(b\right)\right| \text{ if } \left|f^{(n)}\left(a\right)\right|=\left|f^{(n)}\left(b\right)\right|\\ \frac{(b-a)^{n}}{(p\alpha+pn-p+1)^{\frac{1}{p}}}\left(\frac{\left|f^{(n)}(a)\right|^{q}-\left|f^{(n)}(b)\right|^{q}}{\ln\left|f^{(n)}(a)\right|^{q}-\ln\left|f^{(n)}(b)\right|^{q}}\right)^{\frac{1}{q}}\\ \text{if } \left|f^{(n)}\left(a\right)\right|\neq\left|f^{(n)}\left(b\right)\right|.\end{cases}\end{aligned}\tag{3.24}$$

In the case where n is an odd number and $\left|f^{(n)}\left(a\right)\right|\neq\left|f^{(n)}\left(b\right)\right|$, substituting (3.23) in (3.21) we obtain

$$\begin{aligned}&\Bigg|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}(J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a))\\&\quad-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}\left(a\right)+(-1)^{k}f^{(k)}\left(b\right)\right)\Bigg|\end{aligned}$$

$$\leq \frac{(b-a)^n}{2^{\alpha+n+\frac{1}{p}}}\left(\left(\frac{2^{p\alpha+pn-p+1}-1}{p\alpha+pn-p+1}\right)^{\frac{1}{p}}-\left(\frac{1}{p\alpha+pn-p+1}\right)^{\frac{1}{p}}\right)$$
$$\times\left(\left|f^{(n)}(b)\right|^{\frac{1}{2}}-\left|f^{(n)}(a)\right|^{\frac{1}{2}}\right)$$
$$\times\left(\left(\frac{\left|f^{(n)}(a)\right|^{\frac{q}{2}}-\left|f^{(n)}(b)\right|^{\frac{q}{2}}}{\ln\left|f^{(n)}(a)\right|^{q}-\ln\left|f^{(n)}(b)\right|^{q}}\right)^{\frac{1}{q}}\right). \tag{3.25}$$

In the case where n is odd and $\left|f^{(n)}(a)\right|=\left|f^{(n)}(b)\right|$, substituting (3.23) in (3.19) we obtain

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right)-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|$$
$$\leq \frac{(b-a)^n}{2^{\alpha+n}}\left(\left(\frac{2^{p\alpha+pn-p+1}-1}{p\alpha+pn-p+1}\right)^{\frac{1}{p}}-\left(\frac{1}{p\alpha+pn-p+1}\right)^{\frac{1}{p}}\right)\left|f^{(n)}(b)\right|. \tag{3.26}$$

The desired result follows from (3.20), (3.22), (3.24), (3.25), and (3.26). □

Corollary 2. *In Theorem 2 if we take $n=r=1$, we obtain the following fractional trapezoid inequality for differentiable convex functions*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right)-\frac{f(a)+f(b)}{2}\right|$$
$$\leq \frac{b-a}{2^{\alpha+2}}\left(\left(\frac{2^{p\alpha+1}-1}{p\alpha+1}\right)^{\frac{1}{p}}-\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\right)$$
$$\times\left(\left(\frac{|f'(a)|^q+3\left|f^{(n)}(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right).$$

Moreover if we choose $\alpha = 1$, *we get*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}\right| \leq \frac{b-a}{8}\left(\left(\frac{2^{p+1}-1}{p+1}\right)^{\frac{1}{p}} - \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\right)$$
$$\times\left(\left(\frac{|f'(a)|^q+3\left|f^{(n)}(b)\right|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right).$$

Corollary 3. *In Theorem 2 if we take* $n = 1$ *and* $r = 0$, *we obtain the following fractional trapezoid inequality for differentiable* log-*convex functions is given by the following inequality*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)\right) - \frac{f(a)+f(b)}{2}\right|$$
$$\leq \begin{cases} \frac{b-a}{2^{\alpha+1}}\left(\left(\frac{2^{p\alpha+1}-1}{p\alpha+1}\right)^{\frac{1}{p}} - \left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\right)\left|f^{(n)}(b)\right| \\ \quad \text{if } |f'(a)| = |f'(b)|, \\ \frac{b-a}{2^{\alpha+1+\frac{1}{p}}}\left(\left(\frac{2^{p\alpha+1}-1}{p\alpha+1}\right)^{\frac{1}{p}} - \left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\right) \\ \times\left(|f'(b)|^{\frac{1}{2}} - |f'(a)|^{\frac{1}{2}}\right) \\ \times\left(\left(\frac{|f'(a)|^{\frac{q}{2}}-|f'(b)|^{\frac{q}{2}}}{\ln|f'(a)|^q-\ln|f'(b)|^q}\right)^{\frac{1}{q}}\right) \text{ if } |f'(a)| \neq |f'(b)|. \end{cases}$$

Theorem 3. *Let* $f : [a,b] \subset \mathbb{R}_0 \to \mathbb{R}$ *be an* n*-times differentiable function on* (a,b) *such that* $f^{(n)} \in L([a,b])$ *with* $f^{(n)}(a) \neq 0$ *and* $f^{(n)}(b) \neq 0$.*If* $\left|f^{(n)}\right|^q$ *is* r*-convex with* $r \geq 0$, *such that* $q \geq 1$, *then the following fractional inequalities holds*

If n *is an odd number and* $r > 0$

$$\begin{aligned}
&\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}(J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a))\right.\\
&\left.-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right)\right|\\
\leq\quad &\tfrac{(b-a)^{n}(c_{r})^{\frac{1}{q}}}{2}\left[\left(\tfrac{2^{\alpha+n}-1}{(\alpha+n)2^{\alpha+n}}\right)^{1-\frac{1}{q}}\right.\\
&\times\left(\left(\tfrac{2^{\alpha+n+\frac{1}{r}}-1}{(\alpha+n+\frac{1}{r})2^{\alpha+n+\frac{1}{r}}}\left|f^{(n)}(a)\right|^{q}+\Psi\left|f^{(n)}(b)\right|^{q}\right)^{\frac{1}{q}}\right.\\
&+\left(B_{\frac{1}{2}}\left(\alpha+n,\tfrac{1}{r}+1\right)\left|f^{(n)}(a)\right|^{q}\right.\\
&\left.\left.+\tfrac{2^{\alpha+n+\frac{1}{r}}-1}{(\alpha+n+\frac{1}{r})2^{\alpha+n+\frac{1}{r}}}\left|f^{(n)}(b)\right|^{q}\right)^{\frac{1}{q}}\right)\\
&-\left(\tfrac{1}{(\alpha+n)2^{\alpha+n}}\right)^{1-\frac{1}{q}}\\
&\times\left(\left(\Psi\left|f^{(n)}(a)\right|^{q}+\tfrac{1}{(\alpha+n+\frac{1}{r})2^{\alpha+n+\frac{1}{r}}}\left|f^{(n)}(b)\right|^{q}\right)^{\frac{1}{q}}\right.\\
&+\left(\tfrac{1}{(\alpha+n+\frac{1}{r})2^{\alpha+n+\frac{1}{r}}}\left|f^{(n)}(a)\right|^{q}\right.\\
&\left.\left.\left.+B_{\frac{1}{2}}\left(\alpha+n,\tfrac{1}{r}+1\right)\left|f^{(n)}(b)\right|^{q}\right)^{\frac{1}{q}}\right)\right],
\end{aligned}\tag{3.27}$$

If n *is an even number and* $r > 0$

$$\begin{aligned}
&\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}(J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a))\right.\\
&\left.-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right)\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(b-a)^{n}(c_r)^{\frac{1}{q}}}{2(\alpha+n)^{1-\frac{1}{q}}}\Bigg[\left(\frac{1}{\alpha+n+\frac{1}{r}}\left|f^{(n)}(a)\right|^{q}\right.\\
&\quad\left.+B\left(\alpha+n,\tfrac{1}{r}+1\right)\left|f^{(n)}(b)\right|^{q}\right)^{\frac{1}{q}}\\
&\quad+\left(B\left(\tfrac{1}{r}+1,\alpha+n\right)\left|f^{(n)}(a)\right|^{q}\right.\\
&\quad\left.+\frac{1}{\alpha+n+\frac{1}{r}}\left|f^{(n)}(b)\right|^{q}\right)^{\frac{1}{q}}\Bigg],
\end{aligned} \tag{3.28}$$

if $r=0$ *and* $\left|f^{(n)}(a)\right|=\left|f^{(n)}(b)\right|$

$$\begin{aligned}
&\Bigg|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}(J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a))\\
&-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right)\Bigg|\\
\leq&\begin{cases}(b-a)^{n}\left(\frac{2^{\alpha+n-1}-1}{(\alpha+n)2^{\alpha+n-1}}\right)\left|f^{(n)}(b)\right| & \text{if } n \text{ is odd}\\ \frac{(b-a)^{n}}{\alpha+n}\left|f^{(n)}(b)\right| & \text{if } n \text{ is even}\end{cases}
\end{aligned}$$

if $r=0$ *and* $\left|f^{(n)}(a)\right|\neq\left|f^{(n)}(b)\right|$ *and* n *is an odd number*

$$\begin{aligned}
&\Bigg|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}(J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a))\\
&-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right)\Bigg|\\
\leq&\ \frac{(b-a)^{n}\left|f^{(n)}(b)\right|}{2}\Bigg[\left(\frac{2^{\alpha+n}-1}{(\alpha+n)2^{\alpha+n}}\right)^{1-\frac{1}{q}}\\
&\times\Bigg(\Bigg(\sum_{i=1}^{\infty}\frac{(\ln\eta)^{i-1}}{(\alpha+n)_{i}}-\frac{1}{2^{\alpha+n}}\eta^{\frac{1}{2}}\sum_{i=1}^{\infty}\frac{\left(\frac{1}{2}\ln\eta\right)^{i-1}}{(\alpha+n)_{i}}\Bigg)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\eta\sum_{i=1}^{\infty}\frac{(-\ln\eta)^{i-1}}{(\alpha+n)_i}-\frac{1}{2^{\alpha+n}}\eta^{\frac{1}{2}}\sum_{i=1}^{\infty}\frac{\left(-\frac{1}{2}\ln\eta\right)^{i-1}}{(\alpha+n)_i}\right)^{\frac{1}{q}}\Bigg)$$

$$-\left(\frac{1}{(\alpha+n)2^{\alpha+n}}\right)^{1-\frac{1}{q}}$$

$$\times\left(\left(\frac{1}{2^{\alpha+n}}\eta^{\frac{1}{2}}\sum_{i=1}^{\infty}\frac{\left(-\frac{1}{2}\ln\eta\right)^{i-1}}{(\alpha+n)_i}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{1}{2^{\alpha+n}}\eta^{\frac{1}{2}}\sum_{i=1}^{\infty}\frac{\left(\frac{1}{2}\ln\eta\right)^{i-1}}{(\alpha+n)_i}\right)^{\frac{1}{q}}\right)\Bigg],$$

if $r=0$ *and* $\left|f^{(n)}(a)\right|\neq\left|f^{(n)}(b)\right|$ *and* n *is an even number*

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right)\right.$$

$$\left.-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|$$

$$\leq\frac{(b-a)^n\left|f^{(n)}(b)\right|}{2(\alpha+n)^{1-\frac{1}{q}}}\left(\left(\sum_{i=1}^{\infty}\frac{(\ln\eta)^{i-1}}{(\alpha+n)_i}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\eta\sum_{i=1}^{\infty}\frac{(-\ln\eta)^{i-1}}{(\alpha+n)_i}\right)^{\frac{1}{q}}\right). \tag{3.29}$$

where

$$\Psi=B\left(\alpha+n,\frac{1}{r}+1\right)-B_{\frac{1}{2}}\left(\alpha+n,\frac{1}{r}+1\right), \tag{3.30}$$

$$\eta=\frac{\left|f^{(n)}(a)\right|^q}{\left|f^{(n)}(b)\right|^q}, \tag{3.31}$$

c_r *is defined as in* (3.1), $B(.,.)$ *and* $B_x(.,.)$ *are beta and incomplete beta functions respectively.*

Proof. From Lemma 3, and properties of modulus, we have

$$\begin{aligned}
&\Big|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}(J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)) \\
&-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\Big| \\
\leq\ & \tfrac{(b-a)^n}{2}\int_0^1\left|(-1)^{n-1}(1-t)^{\alpha+n-1}-t^{\alpha+n-1}\right| \\
&\left|f^{(n)}(at+(1-t)b)\right|dt \\
=\ & \begin{cases}
\tfrac{(b-a)^n}{2}\int\limits_0^1\left((1-t)^{\alpha+n-1}+t^{\alpha+n-1}\right) \\
\left|f^{(n)}(at+(1-t)b)\right|dt \text{ if } n \text{ even} \\
\tfrac{(b-a)^n}{2}\left(\int\limits_0^{\frac{1}{2}}\left((1-t)^{\alpha+n-1}-t^{\alpha+n-1}\right)\right. \\
\left|f^{(n)}(at+(1-t)b)\right|dt \\
+\int\limits_{\frac{1}{2}}^1\left(t^{\alpha+n-1}-(1-t)^{\alpha+n-1}\right) \\
\left.\times\left|f^{(n)}(at+(1-t)b)\right|dt\right) \\
\text{if } n \text{ odd.}
\end{cases}
\end{aligned} \tag{3.32}$$

Assume that $r>0$ and n is an even number.

From (3.32), power mean inequality, r-convexity of $\left|f^{(n)}\right|^q$, and Lemma 1, we have

$$\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}(J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a))\right.$$

$$
\begin{aligned}
& -\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\Bigg| \\
\leq\ & \frac{(b-a)^n}{2}\left(\int_0^1 t^{\alpha+n-1}dt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1 t^{\alpha+n-1}\left|f^{(n)}(at+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\
& +\left(\int_0^1 (1-t)^{\alpha+n-1}dt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1 (1-t)^{\alpha+n-1}\left|f^{(n)}(at+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{(b-a)^n(c_r)^{\frac{1}{q}}}{2(\alpha+n)^{1-\frac{1}{q}}}\Bigg[\Bigg(\left|f^{(n)}(a)\right|^q\int_0^1 t^{\alpha+n+\frac{1}{r}-1}dt \\
& +\left|f^{(n)}(b)\right|^q\int_0^1 t^{\alpha+n-1}(1-t)^{\frac{1}{r}}dt\Bigg)^{\frac{1}{q}} \\
& +\Bigg(\left|f^{(n)}(a)\right|^q\int_0^1 t^{\frac{1}{r}}(1-t)^{\alpha+n-1}dt \\
& +\left|f^{(n)}(b)\right|^q\int_0^1 (1-t)^{\alpha+n+\frac{1}{r}-1}dt\Bigg)^{\frac{1}{q}}\Bigg] \\
=\ & \frac{(b-a)^n(c_r)^{\frac{1}{q}}}{2(\alpha+n)^{1-\frac{1}{q}}}\Bigg(\Bigg(\frac{1}{\alpha+n+\frac{1}{r}}\left|f^{(n)}(a)\right|^q
\end{aligned}
$$

$$+B\left(\alpha+n,\tfrac{1}{r}+1\right)\left|f^{(n)}(b)\right|^{q}\Big)^{\frac{1}{q}}$$
$$+\left(B\left(\tfrac{1}{r}+1,\alpha+n\right)\left|f^{(n)}(a)\right|^{q}+\tfrac{1}{\alpha+n+\frac{1}{r}}\left|f^{(n)}(b)\right|^{q}\right)^{\frac{1}{q}}\Bigg), \tag{3.33}$$

where c_r is defined by (3.1).

In the case where $r>0$ and n is an odd number, using (3.32), power mean inequality, r-convexity of $\left|f^{(n)}\right|^{q}$, and Lemma 1 we obtain

$$\begin{aligned}
&\Big|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right)\\
&-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right)\Big|\\
\leq\ &\tfrac{(b-a)^{n}}{2}\Bigg(\Bigg[\left(\int_{0}^{\frac{1}{2}}(1-t)^{\alpha+n-1}\right)^{1-\frac{1}{q}}\\
&\times\left(\int_{0}^{\frac{1}{2}}(1-t)^{\alpha+n-1}\left|f^{(n)}(at+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\Bigg]\\
&-\Bigg[\left(\int_{0}^{\frac{1}{2}}t^{\alpha+n-1}\right)^{1-\frac{1}{q}}\\
&\times\left(\int_{0}^{\frac{1}{2}}t^{\alpha+n-1}\left|f^{(n)}(at+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\Bigg]
\end{aligned}$$

$$+\left[\left(\int_{\frac{1}{2}}^{1} t^{\alpha+n-1}\right)^{1-\frac{1}{q}}\right.$$

$$\left.\times\left(\int_{\frac{1}{2}}^{1} t^{\alpha+n-1}\left|f^{(n)}\left(at+(1-t)\,b\right)\right|^{q} dt\right)^{\frac{1}{q}}\right]$$

$$-\left[\left(\int_{\frac{1}{2}}^{1} (1-t)^{\alpha+n-1}\right)^{1-\frac{1}{q}}\right.$$

$$\left.\left.\times\left(\int_{\frac{1}{2}}^{1} (1-t)^{\alpha+n-1}\left|f^{(n)}\left(at+(1-t)\,b\right)\right|^{q} dt\right)^{\frac{1}{q}}\right]\right)$$

$$= \frac{(b-a)^{n}(c_r)^{\frac{1}{q}}}{2}\left[\left(\frac{2^{\alpha+n}-1}{(\alpha+n)2^{\alpha+n}}\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\left(\left|f^{(n)}\left(a\right)\right|^{q}\int_{\frac{1}{2}}^{1} t^{\alpha+n+\frac{1}{r}-1}dt+\left|f^{(n)}\left(b\right)\right|^{q}\right.\right.$$

$$\left.\times\int_{\frac{1}{2}}^{1} t^{\alpha+n-1}\left(1-t\right)^{\frac{1}{r}} dt\right)^{\frac{1}{q}}$$

$$+\left(\left|f^{(n)}\left(a\right)\right|^{q}\int_{0}^{\frac{1}{2}} t^{\frac{1}{r}}\left(1-t\right)^{\alpha+n-1} dt\right.$$

$$
\begin{aligned}
&+\left|f^{(n)}(b)\right|^q \int_0^{\frac{1}{2}} (1-t)^{\alpha+n+\frac{1}{r}-1}\,dt\Bigg)^{\frac{1}{q}}\Bigg)\\
&-\left(\tfrac{1}{(\alpha+n)2^{\alpha+n}}\right)^{1-\frac{1}{q}}\\
&\times\Bigg(\Bigg(\left|f^{(n)}(a)\right|^q \int_0^{\frac{1}{2}} t^{\alpha+n+\frac{1}{r}-1}dt\\
&+\left|f^{(n)}(b)\right|^q \int_0^{\frac{1}{2}} t^{\alpha+n-1}(1-t)^{\frac{1}{r}}\,dt\Bigg)^{\frac{1}{q}}\\
&+\Bigg(\left|f^{(n)}(a)\right|^q \int_{\frac{1}{2}}^{1} t^{\frac{1}{r}}(1-t)^{\alpha+n-1}\,dt\\
&+\left|f^{(n)}(b)\right|^q \int_{\frac{1}{2}}^{1} (1-t)^{\alpha+n+\frac{1}{r}-1}\,dt\Bigg)^{\frac{1}{q}}\Bigg)\Bigg]\\
=\ &\tfrac{(b-a)^n (c_r)^{\frac{1}{q}}}{2}\Bigg[\left(\tfrac{2^{\alpha+n}-1}{(\alpha+n)2^{\alpha+n}}\right)^{1-\frac{1}{q}}\\
&\times\Bigg(\Bigg(\tfrac{2^{\alpha+n+\frac{1}{r}}-1}{\left(\alpha+n+\frac{1}{r}\right)2^{\alpha+n+\frac{1}{r}}}\left|f^{(n)}(a)\right|^q+\Psi\left|f^{(n)}(b)\right|^q\Bigg)^{\frac{1}{q}}\\
&+\Bigg(B_{\frac{1}{2}}\left(\alpha+n,\tfrac{1}{r}+1\right)\left|f^{(n)}(a)\right|^q\\
&+\tfrac{2^{\alpha+n+\frac{1}{r}}-1}{\left(\alpha+n+\frac{1}{r}\right)2^{\alpha+n+\frac{1}{r}}}\left|f^{(n)}(b)\right|^q\Bigg)^{\frac{1}{q}}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\tfrac{1}{(\alpha+n)2^{\alpha+n}}\right)^{1-\frac{1}{q}}\\
&\times\left(\left(\Psi\left|f^{(n)}(a)\right|^{q}+\tfrac{1}{(\alpha+n+\frac{1}{r})2^{\alpha+n+\frac{1}{r}}}\left|f^{(n)}(b)\right|^{q}\right)^{\frac{1}{q}}\right.\\
&+\left(\tfrac{1}{(\alpha+n+\frac{1}{r})2^{\alpha+n+\frac{1}{r}}}\left|f^{(n)}(a)\right|^{q}\right.\\
&\left.\left.+B_{\frac{1}{2}}\left(\alpha+n,\tfrac{1}{r}+1\right)\left|f^{(n)}(b)\right|^{q}\right)^{\frac{1}{q}}\right)\Bigg],
\end{aligned}
\tag{3.34}
$$

where c_r and Ψ are defined as in (3.1) and (3.30) respectively.

Now, we assume that $r=0$, then $\left|f^{(n)}\right|^{q}$ is log-convex, we have

$$
\begin{aligned}
&\left|f^{(n)}(at+(1-t)b)\right|^{q}\le\left(\left|f^{(n)}(a)\right|^{q}\right)^{t}\left(\left|f^{(n)}(b)\right|^{q}\right)^{1-t}\\
&=\left|f^{(n)}(b)\right|^{q}\left(\tfrac{\left|f^{(n)}(a)\right|^{q}}{\left|f^{(n)}(b)\right|^{q}}\right)^{t}.
\end{aligned}
\tag{3.35}
$$

In the case where n is an even number, from (3.32) and (3.35), and power mean inequality, we get

$$
\begin{aligned}
&\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right)\right.\\
&\left.-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right)\right|\\
\le\;&\tfrac{(b-a)^{n}\left|f^{(n)}(b)\right|}{2(\alpha+n)^{1-\frac{1}{q}}}\left(\left(\int_{0}^{1}(1-t)^{\alpha+n-1}\eta^{t}dt\right)^{\frac{1}{q}}\right.
\end{aligned}
$$

$$+\left(\int_0^1 t^{\alpha+n-1}\eta^t dt\right)^{\frac{1}{q}}\Bigg), \tag{3.36}$$

where η is defined by (3.31).

If $\eta = 1$, (3.36) gives

$$\begin{aligned}&\Bigg|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}(J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a))\\&-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\Bigg|\\ \leq\ &\tfrac{(b-a)^n\left|f^{(n)}(b)\right|}{(\alpha+n)}.\end{aligned} \tag{3.37}$$

And if $\eta \neq 1$, from (3.36) and Lemma 2 we deduce

$$\begin{aligned}&\Bigg|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}(J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a))\\&-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\Bigg|\\ \leq\ &\tfrac{(b-a)^n\left|f^{(n)}(b)\right|}{2(\alpha+n)^{1-\frac{1}{q}}}\left(\left(\sum_{i=1}^{\infty}\tfrac{(\ln\eta)^{i-1}}{(\alpha+n)_i}\right)^{\frac{1}{q}}\right.\\&\left.+\left(\eta\sum_{i=1}^{\infty}\tfrac{(-\ln\eta)^{i-1}}{(\alpha+n)_i}\right)^{\frac{1}{q}}\right),\end{aligned} \tag{3.38}$$

where η is defined by (3.31).

In the case where n is an odd number and $\eta = 1$, from (3.32) and (3.35), and power mean inequality, we get

$$\begin{aligned}
&\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a+}^{\alpha} f(b)+J_{b-}^{\alpha} f(a)\right)\right. \\
&\left.-\sum_{k=0}^{n-1} \tfrac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k} f^{(k)}(b)\right)\right| \\
\leq\ & (b-a)^{n}\left(\tfrac{2^{\alpha+n-1}-1}{(\alpha+n)2^{\alpha+n-1}}\right)\left|f^{(n)}(b)\right|. \qquad (3.39)
\end{aligned}$$

In the case where n is an odd number and $\eta \neq 1$, substituting (3.32) and (3.35), and power mean inequality, we get

$$\begin{aligned}
&\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a+}^{\alpha} f(b)+J_{b-}^{\alpha} f(a)\right)\right. \\
&\left.-\sum_{k=0}^{n-1} \tfrac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k} f^{(k)}(b)\right)\right| \\
\leq\ & \tfrac{(b-a)^{n}\left|f^{(n)}(b)\right|}{2} \\
&\times\left(\left(\tfrac{2^{\alpha+n}-1}{(\alpha+n)2^{\alpha+n}}\right)^{1-\frac{1}{q}}\left(\left(\int_{0}^{\frac{1}{2}}(1-t)^{\alpha+n-1}\eta^{t}dt\right)^{\frac{1}{q}}\right.\right. \\
&\left.+\left(\int_{\frac{1}{2}}^{1} t^{\alpha+n-1}\eta^{t}dt\right)^{\frac{1}{q}}\right) \\
&-\left(\tfrac{1}{(\alpha+n)2^{\alpha+n}}\right)^{1-\frac{1}{q}}\left(\left(\int_{0}^{\frac{1}{2}} t^{\alpha+n-1}\eta^{t}dt\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(\int_{\frac{1}{2}}^{1}(1-t)^{\alpha+n-1}\eta^{t}dt\right)^{\frac{1}{q}}\Bigg)\Bigg),$$

where η is defined by (3.31).

Using in the above inequality (3.14)-(3.17) by replacing $\left|f^{(n)}(a)\right|$ by $\left|f^{(n)}(a)\right|^{q}$ and $\left|f^{(n)}(b)\right|$ by $\left|f^{(n)}(b)\right|^{q}$, we get

$$\begin{aligned}
&\Bigg|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right) \\
&-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right)\Bigg| \\
\leq\;& \tfrac{(b-a)^{n}\left|f^{(n)}(b)\right|}{2}\Bigg[\left(\tfrac{2^{\alpha+n}-1}{(\alpha+n)2^{\alpha+n}}\right)^{1-\frac{1}{q}} \\
&\times\left(\left(\sum_{i=1}^{\infty}\tfrac{(\ln\eta)^{i-1}}{(\alpha+n)_{i}}-\tfrac{1}{2^{\alpha+n}}\eta^{\frac{1}{2}}\sum_{i=1}^{\infty}\tfrac{\left(\frac{1}{2}\ln\eta\right)^{i-1}}{(\alpha+n)_{i}}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\eta\sum_{i=1}^{\infty}\tfrac{(-\ln\eta)^{i-1}}{(\alpha+n)_{i}}-\tfrac{1}{2^{\alpha+n}}\eta^{\frac{1}{2}}\sum_{i=1}^{\infty}\tfrac{\left(-\frac{1}{2}\ln\eta\right)^{i-1}}{(\alpha+n)_{i}}\right)^{\frac{1}{q}}\right) \\
&-\left(\tfrac{1}{(\alpha+n)2^{\alpha+n}}\right)^{1-\frac{1}{q}} \\
&\times\left(\left(\tfrac{1}{2^{\alpha+n}}\eta^{\frac{1}{2}}\sum_{i=1}^{\infty}\tfrac{\left(-\frac{1}{2}\ln\eta\right)^{i-1}}{(\alpha+n)_{i}}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\tfrac{1}{2^{\alpha+n}}\eta^{\frac{1}{2}}\sum_{i=1}^{\infty}\tfrac{\left(\frac{1}{2}\ln\eta\right)^{i-1}}{(\alpha+n)_{i}}\right)^{\frac{1}{q}}\right)\Bigg]. \qquad (3.40)
\end{aligned}$$

Thus the desired results follows from (3.33), (3.34), (3.37), (3.38), (3.39), and (3.40). The Proof is completed. □

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